Equivariant Covering Spaces of Quantum Homogeneous Spaces

Mao HOSHINO
(University of Tokyo)

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My latest work (arXiv: 2301.04975)

Some fundamental facts on G-equivariant inclusions

- Algebraic & analytic characterizations of fin. index inclusions
- The range of covering degrees.

Imprimitivity results for equivariant correspondences

- G_q where G : a 1-connected compact Lie group.
- Quantum homogeneous spaces with a finiteness condition.

Equivariant quantum covering space

$A \overset{E}{\curvearrowleft} \subset B$: of fin index $\overset{\text{def}}{\Longleftrightarrow}$ $\exists (v_i)_{i=1}^n \subset B$ s.t. $b = \sum_{i=1}^n v_i E(v_i^* b)$

$\mathrm{Index}\, E := \sum_{i=1}^n v_i v_i^*$ ← independent of $(v_i)_{i=1}^n$

<u>Def</u>

$\mathbb{G}$: a comapct quantum group.

A : a unital C^*-algebra with a $\mathbb{G}$-action.

quantum $\mathbb{G}$- covering space ⋯ $A \subset B$: $\mathbb{G}$-equiv. unital inclusion

$\exists E$: $\mathbb{G}$-equiv , of fin index.

covering degree $= \inf_{E : \mathbb{G}\text{-equiv}} \| \mathrm{Index}\, E \|$

Example : covering space

X, Y : compact Hausdorff G-sp

$\pi : Y \longrightarrow X$: a G-equivariant continuous surjection.

$\rightsquigarrow \pi^* : C(X) \longrightarrow C(Y)$: a G-equivariant unital inclusion

Fact

$$\boxed{\begin{array}{l} \pi : \text{covering map} \iff {}^{\exists}E : C(Y) \longrightarrow C(X) : \text{of fin. index} \\ \quad \cdot\ E(f)(x) = \dfrac{1}{|\pi^{-1}(x)|} \displaystyle\sum_{y \in \pi^{-1}(x)} f(y) \\ \quad \cdot\ \mathrm{Index}\, E = \text{covering degree.} \end{array}}$$

unital inclusion of C^*-alg admitting a fin. index cond. exp "=" Noncommutative covering sp.

Other examples

- $\Lambda \leq \Gamma$: an inclusion of discrete groups

$\rightsquigarrow$ $C_r^*(\Lambda) \subset C_r^*(\Gamma)$ ($\hat{\Gamma}$-equivariant)

$\exists !$ $E : C_r^*(\Gamma) \to C_r^*(\Lambda) : \hat{\Gamma}$-equiv $\quad E\left(\sum_{g \in \Gamma} a_g g\right) = \sum_{g \in \Lambda} a_g g$

E : of fin index $\iff [\Gamma : \Lambda] < \infty$

$\mathrm{Index}\, E = [\Gamma : \Lambda]$

- $\pi = (H_\pi, U_\pi)$: fin. dim'l unitary rep'n of $\mathbb{G}$

$\rightsquigarrow$ $\mathbb{C} \subset \mathcal{B}(H_\pi)$ with $\mathbb{G} \curvearrowright \mathcal{B}(H_\pi)$: adjoint

the covering degree $= (\dim_q \pi)^2$

Equivariant correspondence

$\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$

Def

G-equiv. (A, B)-corr

$\ldots \left\{ \begin{array}{l} M : \text{Hilbert } B\text{-mod} \\ \tilde{\beta} : G \curvearrowright M \\ A \longrightarrow \mathcal{L}_B(M) \end{array} \right.$ with $\left\{ \begin{array}{l} \langle \tilde{\beta}_g(x), \tilde{\beta}_g(y) \rangle_B = \beta_g(\langle x, y \rangle_B) \\ \tilde{\beta}_g(xb) = \tilde{\beta}_g(x)\beta_g(b) \\ \tilde{\beta}_g(ax) = \alpha_g(a)\tilde{\beta}_g(x) \end{array} \right.$

G-equiv Hilbert B-mod = G-equiv $(\mathbb{C}, B)$-corr.

$G\text{-Mod}^f_A$: the cat of fin. generated G-equiv. Hilb A-mod.

$G\text{-Corr}^{rf}_{A,B}$: the cat of right fin. generated G-equiv. (A, B)-corr.

Q-systems in $G\text{-}\mathrm{Corr}_A^{rf}$

$A \subset B$: quantum G-covering space with $A^G = \mathbb{C}1_A$

E : G-equiv cond. exp of fin. index.

$\rightsquigarrow$ B_E = B as an A-bimodule

$\langle x, y \rangle_A = E(x^* y) \quad (x, y \in A)$ $\Big\} \longrightarrow$ G-equiv. corr. $/A$

- $m : B_E \otimes_A B_E \longrightarrow B_E \,;\, x \otimes y \longmapsto xy$
- $u : A \longrightarrow B_E \,;\, a \longmapsto a$

semisimple C^*-tensor cat (not multitensor)

(B_E, m, u) : C^*- Frobenius alg. in $G\text{-}\mathrm{Corr}_A^{rf}$

$\}$ normalization

Q-system

Thm

- $\{ A \subset B : \text{quantum } \mathbb{G}\text{-covering space} \} / \text{isom}$

$$\overset{1:1}{\longleftrightarrow} \quad \{ \text{Q-system in } \mathbb{G}\text{-Cov}_A^{\text{rf}} \} / \text{isom}$$

- the covering degree of $A \subset B$

$$= \text{the categorical dim. of } B_E \text{ in } \mathbb{G}\text{-Cov}_A^{\text{rf}}$$

e.g. $\mathbb{C} \subset B(\mathcal{H}_\pi)$: the adjoint action

$$\rightsquigarrow \mathbb{G}\text{-Cov}_{\mathbb{C}}^{\text{rf}} = \text{Rep}^f \mathbb{G} \quad \& \quad B_E \simeq \pi \otimes \overline{\pi}$$

$\therefore$ the covering degree $= (\dim_q \pi)^2$

Tannaka-Krein duality

$\alpha : \mathbb{G} \curvearrowright A$ s.t. $A^{\mathbb{G}} = \mathbb{C} 1_A$

$$\begin{cases} \pi \in \mathrm{Rep}^f \mathbb{G} \\ E \in \mathbb{G}\text{-}\mathrm{Mod}_A^f \end{cases} \rightsquigarrow H_\pi \otimes E \in \mathbb{G}\text{-}\mathrm{Mod}_A^f$$

- $\tilde{\alpha}_g(\xi \otimes x) = \pi(g)\xi \otimes \tilde{\alpha}_g(x)$
- $\langle \xi \otimes x, \eta \otimes y \rangle_A = \langle \xi, \eta \rangle \langle x, y \rangle_A$

$\rightsquigarrow$ This makes $\mathbb{G}\text{-}\mathrm{Mod}_A^f$ into a left $\mathrm{Rep}^f \mathbb{G}$-module category.

<u>Thm</u> (De Commer-Yamashita, Neshveyev)

$$\boxed{\begin{array}{c} \{\text{quantum homogeneous spaces of } \mathbb{G}\} \big/ \mathbb{G}\text{-equiv. Morita equiv.} \\ \overset{1:1}{\longleftrightarrow} \{\text{connected left } \mathrm{Rep}^f \mathbb{G}\text{-module category}\} \big/ \text{equiv.} \end{array}}$$

$G\text{-}\mathrm{Corr}_A^{f} \leftarrow$ depends only on G-equiv. Morita equiv. class of A !

Thm

$\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$

$\rightsquigarrow$ $G\text{-}\mathrm{Corr}_{A,B}^{rf} \simeq [G\text{-}\mathrm{Mod}_A^f , G\text{-}\mathrm{Mod}_B^f]^{\mathrm{Rep}^f G}$

the category of $\mathrm{Rep}^f G$ - module functors

$\mathrm{Rep}^f G$- module functor . . .
$\left\{ \begin{array}{l} F : G\text{-}\mathrm{Mod}_A^f \longrightarrow G\text{-}\mathrm{Mod}_B^f \\ \{ f : F(\mathcal{H}_\pi \otimes E) \xrightarrow{\simeq} \mathcal{H}_\pi \otimes F(E) \}_{\pi, E} \end{array} \right.$

e.g. $M \in G\text{-}\mathrm{Corr}_{A,B}^{rf} \rightsquigarrow F(E) := E \underset{A}{\otimes} M$

$f : (\mathcal{H}_\pi \otimes E) \underset{A}{\otimes} M \simeq \mathcal{H}_\pi \otimes (E \underset{A}{\otimes} M)$

Proof of Thm

Thm is a generalization of the following:

Thm (De Commer - Yamashita)

$$\{ G\text{-equiv } *\text{-hom from } A \text{ to } B \} \overset{1:1}{\longleftrightarrow} \left\{ (F, f) \in [G\text{-Mod}_A^f, G\text{-Mod}_B^f]^{\mathrm{Rep}^f G} \ \text{s.t. } F(A) = B \right\} \Big/ \substack{\text{natural} \\ \text{equiv.}}$$

$\longrightarrow$ By imitating their proof, we can show the duality theorem for equivariant correspondences.

The range of covering degree

<u>Cor</u>

$$\boxed{\left\{ d \;\middle|\; {}^{\forall}G\ {}^{\forall}A \subset B \text{ s.t } A^{G} = \mathbb{C}1_A \right\} = \left\{ 4\cos^2\frac{\pi}{n} \;\middle|\; n \geq 3 \right\} \sqcup [4, \infty)}$$

prf $\underline{\underline{d \geq 4}}$ $\rightsquigarrow$ $\mathbb{C} \subset B(\mathcal{H}_\pi)$

$\underline{\underline{d < 4}}$ Fix $\mathcal{C}$: fusion cat. & (Q, m, u) : Q-system in $\mathcal{C}$ s.t $\dim Q = d$.

$\rightsquigarrow$ ${}^{\exists}\mathrm{Rep}^f SU_q(2) \xrightarrow{\Phi} \mathcal{C}$: "surjective"

Now $\mathrm{Rep}^f SU_q(2) \curvearrowright \mathcal{C}$ by $\pi \otimes X := \Phi(\pi) \underset{\mathcal{C}}{\otimes} X$

$$\therefore\ {}^{\exists}SU_q(2)\text{-Mod}_A^f \simeq \mathcal{C} \rightsquigarrow \begin{array}{ccc} \mathcal{C}^\vee & \to & [SU_q(2)\text{-Mod}_A^f,\ SU_q(2)\text{-Mod}_A^f]^{\mathrm{Rep}^f SU_q(2)} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ Q & \longmapsto & B_E \end{array}$$

Classification problem

$\hat{\mathbb{H}} \le \hat{\mathbb{G}}$: discrete quantum subgroup s.t. $E : C(\mathbb{G}) \longrightarrow C(\mathbb{H})$: of fin index

$$\rightsquigarrow \quad C(\mathbb{G}) \subseteq \mathcal{L}_{C(\mathbb{H})}(C(\mathbb{G})_E) = C(\mathbb{G})_E \otimes_{C(\mathbb{H})} C(\mathbb{G})$$

: quantum $\mathbb{G}$-covering space $/ C(\mathbb{G})$

Prop

$\mathbb{G}$: compact quantum group, $\mathcal{O}(\mathbb{G})$: the algebra of matrix coeff.

$$\rightsquigarrow \quad \mathrm{Rep}^f \mathcal{O}(\mathbb{G}) \simeq \mathbb{G}\text{-}\mathrm{Cov}^{rf}_{C(\mathbb{G})}$$

$$(\pi, H) \longmapsto H \otimes C(\mathbb{G})$$

with $x \cdot (\xi \otimes y) = (\pi \otimes id)\Delta(x)(\xi \otimes y)$

Maximal Kac quantum subgroup

Thm (Soltan)

$\mathbb{G}$: compact quantum group

$\rightsquigarrow$ $\exists!$ $\mathbb{K} \leq \mathbb{G}$: of Kac type s.t.

$$\begin{tikzcd} \mathbb{H} \arrow[r, dashed, "\exists!"] \arrow[dr, "\forall"'] & \mathbb{K} \arrow[d] \arrow[dl, phantom, "\circlearrowleft" near start] \\ & \mathbb{G} \end{tikzcd}$$

($\mathbb{H}$: of Kac type)

the maximal Kac quantum subgroup.

e.g. (Tomatsu) $\mathbb{G} = G_q$ $(0 < |q| < 1) \Longrightarrow \mathbb{K} = \mathbb{T}$

Prop (Soltan)

$q : \mathcal{O}(\mathbb{G}) \longrightarrow \mathcal{O}(\mathbb{K})$: the canonical map

$\rightsquigarrow$ $q^* : \mathrm{Rep}^\dagger \mathcal{O}(\mathbb{K}) \xrightarrow{\simeq} \mathrm{Rep}^\dagger \mathcal{O}(\mathbb{G})$

$$\rightsquigarrow \quad \begin{tikzcd} \mathrm{Rep}^f \mathcal{O}(\mathbb{K}) \arrow[r, "\simeq"', "q^*"] & \mathrm{Rep}^f \mathcal{O}(\mathbb{G}) \\ \mathbb{K}\text{-}\mathrm{Corr}^{rf}_{C(\mathbb{K})} \arrow[u, phantom, sloped, "\simeq"] \arrow[r, "\mathrm{Ind}_{\mathbb{K}}^{\mathbb{G}}"'] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{G}\text{-}\mathrm{Corr}^{rf}_{C(\mathbb{G})} \arrow[u, phantom, sloped, "\simeq"] \end{tikzcd}$$

e.g. Any quantum G_q-covering space over $C(G_q)$ must be induced from a quantum T-covering space over $C(T)$.

Problem

A, B : quantum homogeneous space of $\mathbb{K}$

$\tilde{A} = \mathrm{Ind}_{\mathbb{K}}^{\mathbb{G}} A$, $\tilde{B} = \mathrm{Ind}_{\mathbb{K}}^{\mathbb{G}} B$.

$$\rightsquigarrow \mathrm{Ind}_{\mathbb{K}}^{\mathbb{G}} : \mathbb{K}\text{-}\mathrm{Corr}^{rf}_{A,B} \xrightarrow{\simeq ?} \mathbb{G}\text{-}\mathrm{Corr}^{rf}_{\tilde{A},\tilde{B}}$$

Thm (H.)

The functor $\mathrm{Ind}_{\mathbb{K}}^{\mathbb{G}} : \mathbb{K}\text{-}\mathrm{Corr}^{rf}_{A,B} \longrightarrow \mathbb{G}\text{-}\mathrm{Corr}^{rf}_{\tilde{A},\tilde{B}}$ is an equivalence

when 1. $\mathbb{G} = G_q$ (the Drinfeld-Jimbo deformation with $0<|q|<1$)

2. $\exists$ tracial states on A, B

& $\mathrm{Irr}\,\mathbb{K}\text{-}\mathrm{Mod}^f_A$, $\mathrm{Irr}\,\mathbb{K}\text{-}\mathrm{Mod}^f_B$ are finite.

If $\mathbb{K}$ is cocommutative, we also have the following partial answer.

3. $\mathrm{Aut}_{\mathbb{K}}(A) \simeq \mathrm{Aut}_{\mathbb{G}}(\tilde{A})$, $\mathrm{Pic}_{\mathbb{K}}(A) \simeq \mathrm{Pic}_{\mathbb{G}}(\tilde{A})$

Cor

- quantum $\mathbb{G}$-covering spaces over $\mathrm{Ind}_{\mathbb{K}}^{\mathbb{G}} A$

  $\overset{1:1}{\longleftrightarrow}$ quantum $\mathbb{K}$-covering spaces over A.

- Finite index discrete quantum subgroups of $\widehat{G_q}$ are classified by subgroups of P/Q ← root lattice

  ↑ weight lattice

(By the theorem, we can show the following:

$$\underset{E}{A \leq C(G_q)} \rightsquigarrow A = q^{-1}(\exists A_0) \quad \left(\begin{array}{l} q : C(G_q) \to C(T) \\ A_0 \subset C(T) \end{array} \right.$$

$\therefore\ {}^{\forall}\hat{H} \leq \hat{G_q}$: fin index $\exists ! \Gamma \leq \hat{T} = P$ s.t.

$$\mathrm{Obj}\,\mathrm{Rep}^f H = \{ \pi \in \mathrm{Rep}^f G_q \mid \mathrm{wt}\,\pi \subseteq \Gamma \}$$

Thm (H.)

The answer is "yes" in the following cases.

1. $\mathbb{G} = G_q$ (the Drinfeld-Jimbo deformation of G)

2. $\exists$ tracial states on A, B & $\mathrm{Irr}\,\mathbb{K}\text{-}\mathrm{Mod}^{f}_{A}$, $\mathrm{Irr}\,\mathbb{K}\text{-}\mathrm{Mod}^{f}_{B}$ are finite.

If $\mathbb{K}$ is cocommutative, we also have the following partial answer.

3. $\mathrm{Aut}_{\mathbb{K}}(A) \simeq \mathrm{Aut}_{\mathbb{G}}(\tilde{A})$, $\mathrm{Pic}_{\mathbb{K}}(A) \simeq \mathrm{Pic}_{\mathbb{G}}(\tilde{A})$

1 $\rightsquigarrow$ representation theoretical approach.

2 & 3 $\rightsquigarrow$ module categorical approach.

For the result 1 $\left(\tilde{A} = \mathrm{Ind}_T^{G_q} A,\ \tilde{B} = \mathrm{Ind}_T^{G_q} B\right)$

$\tilde{M} \in G_q\text{-}\mathrm{Corr}^{rf}_{\tilde{A},\tilde{B}}$

$\rightsquigarrow \exists\, M \in T\text{-}\mathrm{Mod}^f_B$ s.t. $\tilde{M} = \mathrm{Ind}_T^{G_q} M$ in $\underline{G_q\text{-}\mathrm{Mod}^f_{\tilde{B}}}$

$$\mathcal{L}_{\tilde{B}}(\tilde{M}) \simeq \mathrm{Ind}_T^{G_q} \mathcal{L}_B(M)$$

<u>What we have to show</u>

Any $*$-hom from $\tilde{A}$ to $\mathrm{Ind}_T^{G_q}\mathcal{L}_B(M)$ is induced from a $*$-hom from A to $\mathcal{L}_B(M)$.

$$\begin{tikzcd}
\mathrm{Ind}_T^{G_q} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Ind}_T^{G_q} \mathcal{L}_B(M) \arrow[d] & \\
A \arrow[rr, bend right, "\exists\, ?"'] & \mathcal{L}_B(M) \arrow[r, "\text{faithful trace}"] & \mathbb{C}
\end{tikzcd}$$

Lem

A : a unital C^*-alg with a T-action

$\rightsquigarrow$ Any tracial state on $\mathrm{Ind}_T^{G_q} A$ descends to a tracial state on A

- $C(T\backslash G_q)^{**} \simeq \prod_{w\in W} B(\mathcal{H}_w)$ $\leftarrow$ infin. dim'l if $w \neq e$
- $p_e = (1, 0, 0, \ldots, 0) \in C(T\backslash G_q)^{**}$

$\rightsquigarrow$ $p_e \in Z(C(G_q)^{**})$ & $p_e C(G_q)^{**} \simeq C(T)^{**}$

$$\begin{tikzcd}
\mathrm{Ind}_T^{G_q} A \arrow[d] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes C(G_q) \arrow[r, "\tilde{\varphi}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \\
A \arrow[r] & A\otimes C(T) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

$\left(\tilde{\varphi}|_{\mathrm{Ind}_T^{G_q} A} : \text{tracial}\right)$

For the result 2 & 3

Key observation

$$\begin{tikzcd}
\mathbb{K}\text{-}\mathrm{Cov}^{rf}_{A,B} \arrow[r, "\simeq"] \arrow[dd, "\mathrm{Ind}^{\mathbb{G}}_{\mathbb{K}}"'] \arrow[ddr, phantom, "\circlearrowright"] & {[\mathbb{K}\text{-}\mathrm{Mod}^{f}_{A}, \mathbb{K}\text{-}\mathrm{Mod}^{f}_{B}]^{\mathrm{Rep}^f \mathbb{K}}} \arrow[d] \\
 & {[\mathbb{K}\text{-}\mathrm{Mod}^{f}_{A}, \mathbb{K}\text{-}\mathrm{Mod}^{f}_{B}]^{\mathrm{Rep}^f \mathbb{G}}} \arrow[d, "\simeq"] \\
\mathbb{G}\text{-}\mathrm{Cov}^{rf}_{\tilde{A},\tilde{B}} \arrow[r, "\simeq"] & {[\mathbb{G}\text{-}\mathrm{Mod}^{f}_{\tilde{A}}, \mathbb{G}\text{-}\mathrm{Mod}^{f}_{\tilde{B}}]^{\mathrm{Rep}^f \mathbb{G}}}
\end{tikzcd}$$

$\rightsquigarrow$ It is enough to show that

$\mathrm{Rep}^f \mathbb{K}$- module functor $=$ $\mathrm{Rep}^f \mathbb{G}$-module functor.

We assume $A = B$ and set $\mathcal{M} := \mathbb{K}\text{-}\mathrm{Mod}_A^f$

$[\mathcal{M}, \mathcal{M}]$: the category of C^*-functors from $\mathcal{M}$ to $\mathcal{M}$

$$\Phi : \mathrm{Rep}^f \mathbb{K} \longrightarrow [\mathcal{M}, \mathcal{M}] \ ; \ \Phi(\pi) = \mathcal{H}_\pi \otimes -$$

$(F, f) \in [\mathcal{M}, \mathcal{M}]^{\mathrm{Rep}^f \mathbb{K}}$

$$\begin{array}{ccccc} f_{\pi, X} : & F(\mathcal{H}_\pi \otimes X) & \longrightarrow & \mathcal{H}_\pi \otimes F(X) \\ & \| & & \| \\ & F \circ \Phi(\pi)(X) & & \Phi(\pi) \circ F(X) \end{array}$$

$$\rightsquigarrow \ f = \{ f_\pi : F \otimes \Phi(\pi) \longrightarrow \Phi(\pi) \otimes F \}_{\pi \in \mathrm{Rep}^f \mathbb{K}}$$

$\therefore$ $\mathrm{Rep}^f \mathbb{K}$-module functor = unitary half-braiding along Φ

Prop

$\mathcal{C}$: a rigid C^*-tensor category.

$(\Phi, \varphi) : \mathrm{Rep}^f \mathbb{K} \longrightarrow \mathcal{C}$: a dim. preserving C^*-tensor functor.

(Z, u) : a unitary half-braiding along $\Phi \circ \mathrm{Res}^{\mathbb{K}}_{\mathbb{G}}$

i.e. $Z \in \mathcal{C}$ & $\{ u_\pi : Z \otimes \Phi(\pi|_{\mathbb{K}}) \xrightarrow{\simeq} \Phi(\pi|_{\mathbb{K}}) \otimes Z \}_{\pi \in \mathrm{Rep}^f \mathbb{G}}$

$\Rightarrow {}^\exists \{ \tilde{u}_\rho : Z \otimes \Phi(\rho) \xrightarrow{\simeq} \Phi(\rho) \otimes Z \}_{\rho \in \mathrm{Rep}^f \mathbb{K}}$

s.t. $\begin{cases} (Z, \tilde{u}) : \text{a unitary half-braiding along } \Phi. \\ \tilde{u}_{\pi|_{\mathbb{K}}} = u_\pi \text{ for } \pi \in \mathrm{Rep}^f \mathbb{G} \end{cases}$

If A has a tracial state

- $\exists \{ \mathrm{Tr}_X : \mathcal{M}(X) \longrightarrow \mathbb{C} \}_{X \in \mathrm{Obj}\,\mathcal{M}}$
  s.t. $\mathrm{Tr}_X(fg) = \mathrm{Tr}_Y(gf)$, $\mathrm{Tr}_{\pi \otimes X} = \mathrm{Tr}_X \circ (\mathrm{Tr}_\pi \otimes \mathrm{id})$.

- We can define a standardness for solutions in $[\mathcal{M}, \mathcal{M}]$ of the conjugate equations.

- $(R, \overline{R})$: standard in $\mathrm{Rep}^f \mathbb{K}$
  $\Rightarrow (\varphi^* \circ \Phi(R), \varphi^* \circ \Phi(\overline{R}))$: standard in $[\mathcal{M}, \mathcal{M}]$.

Based on these facts & $|\mathrm{Irr}\,\mathcal{M}| < \infty$,
we can apply the previous proposition to $[\mathcal{M}, \mathcal{M}]$ & Φ
though $[\mathcal{M}, \mathcal{M}]$ is a multitensor category. ▨

Thank you for
your attension !!